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GENERALIZED ANALYSIS OF EXPERIMENTAL OBSERVATIONS IN PROBLEMS OF ELASTIC STABILITY

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SUMMARY

A generalized method of analyzing experimental observations in problems of elastic stability is presented in which the initial readings of load and deflection may be taken at any load less than the critical load. This analysis is an extension of a method published by Southwell in 1932, in which it was assumed that the initial readings are taken at zero load.

INTRODUCTION

In reference 1, Southwell presented a method for the analysis of experimental observations in problems of elastic stability. Briefly, the method is concerned with the interpretation of simultaneous readings of load and deflection. As therein presented, the method requires that the initial deflection reading be taken at zero load. In the vicinity of zero load, deflection readings are usually somewhat questionable. The deflection readings are reliable only after enough load has been applied thoroughly to seat the specimen and the loading fixtures. Furthermore, it is not always convenient to take the initial deflection reading at zero load. Something may also happen to render the first few readings valueless and it may not be practicable to repeat them. For use under such circumstances, a more general method has been devised wherein the initial readings may be taken at any load less than the critical load.

The general method of analyzing experimental observations in problems of elastic stability is presented in this paper. Reference 1 should be consulted for a detailed discussion of the use and limitations of this type of analysis.

It may also be worth while to study reference 2 because, as early as 1886, Ayrton and Perry recognized the relationships that exist between load and deflection of initially curved members.

THEORY

Consider the simple strut shown in figure 1. Assume the strut to be initially curved. Then under a load P_1, which is less than the critical load, it will have deflections y_1. These deflections can be accurately represented by the series

$$y_1 = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + \dots\dots$$

$$= \sum_{n=1}^{n=\infty} a_n \sin \frac{n\pi x}{L} \qquad (1)$$

Now under an axial load P, which is greater than P_1 but less than the critical load, the deflections y_1 will have been increased by amounts y_2. If the total deflections from the straight form for axial load P are y, then

$$y = y_1 + y_2 \qquad (2)$$

and the bending moment at any cross section is

$$M = P(y_1 + y_2) \qquad (3)$$

Let the bending moment at any cross section be M_1 when the axial load is P_1. Then

$$M_1 = P_1 y_1 \qquad (4)$$

If M_2 is the increase in the bending moment as the axial load increases from P_1 to P, then

$$\begin{aligned} M_2 &= M - M_1 \\ &= P(y_1 + y_2) - P_1 y_1 \end{aligned} \tag{5}$$

or

$$M_2 = (P - P_1)\, y_1 + P y_2 \tag{6}$$

The deflections y_2 caused by axial loads in excess of P_1 are determined from the differential equation

$$EI \frac{d^2 y_2}{dx^2} = -M_2 \tag{7}$$

Substitution of equation (1) in (6) and equation (6) in (7) gives, after division by EI,

$$\frac{d^2 y_2}{dx^2} + \frac{P}{EI} y_2 = -\left(\frac{P}{EI} - \frac{P_1}{EI}\right) \sum_{n=1}^{n=\infty} a_n \sin \frac{n\pi x}{L} \tag{8}$$

The general solution of this equation is

$$\begin{aligned} y_2 &= A \sin \frac{x}{j} + B \cos \frac{x}{j} \\ &\quad + \alpha \left(1 - \frac{P_1}{P}\right) \sum_{n=1}^{n=\infty} \frac{a_n}{n^2 - \alpha} \sin \frac{n\pi x}{L} \end{aligned} \tag{9}$$

where

$$j = \sqrt{\frac{EI}{P}} \tag{10}$$

$$\alpha = \frac{P}{P_{crit}} \tag{11}$$

$$P_{crit} = \frac{\pi^2 EI}{L^2} \tag{12}$$

In order to satisfy the end conditions $(y_2 = 0,$ for $x = 0$ and for $x = L)$ for any value of j, it follows that the constants of integration, A and B, must each equal zero. Therefore, the deflections y_2 are given by the equation

$$y_2 = y - y_1 = \alpha\left(1 - \frac{P_1}{P}\right)\sum_{n=1}^{n=\infty} \frac{a_n}{n^2 - \alpha} \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{n=\infty} \frac{a_n \sin\frac{n\pi x}{L}}{n^2\,\dfrac{P_{crit} - P_1}{P - P_1} - 1}$$

$$= \frac{a_1 \sin\frac{\pi x}{L}}{\dfrac{P_{crit} - P_1}{P - P_1} - 1} + \frac{a_2 \sin\frac{2\pi x}{L}}{2^2\,\dfrac{P_{crit} - P_1}{P - P_1} - 1} \tag{13}$$

As P approaches P_{crit}, the first term in the series of equation (13) predominates. In this case it is possible to write as an approximation for equation (13)

$$y_2 = y - y_1 = \frac{a_1 \sin\frac{\pi x}{L}}{\dfrac{P_{crit} - P_1}{P - P_1} - 1} \tag{14}$$

But equation (14) can also be written in the form

$$\frac{y - y_1}{P - P_1} = \frac{y - y_1}{P_{crit} - P_1} + \frac{a_1 \sin \frac{\pi x}{L}}{P_{crit} - P_1} \tag{15}$$

It is recalled that $y - y_1$ is the amount by which the deflections are increased when the axial load on the strut is increased from P_1 to P. For any assumed initial load P_1 the difference $P_{crit} - P_1$ is a constant. Also, for any assumed cross section at which the lateral deflections $y - y_1$ are measured, the term $\left[a_1 \sin \frac{\pi x}{L}\right]$ is a constant. Hence, if $\frac{y - y_1}{P - P_1}$ is plotted as ordinate against $y - y_1$ as abscissa (fig. 2), it is recognized that equation (15) is a straight line. This line will cut the horizontal axis at the distance $\left[a_1 \sin \frac{\pi x}{L}\right]$ from the origin and the inverse slope of the line is $P_{crit} - P_1$. Thus if simultaneous readings of axial load and deflection are taken during a column test beginning with any load P_1 as the initial reading and these data are plotted in the manner just described, the reciprocal of the slope of the straight line obtained is the value of $P_{crit} - P_1$. The value of P_{crit} is then obtained from the relation

$$P_{crit} = \left(P_{crit} - P_1\right) + P_1 \tag{16}$$

As mentioned by Southwell in reference 1, the main interest of this method of analysis lies in its generality because, in all ordinary examples of elastic instability, the same type of differential equation governs the deflection as controlled by its initial value, provided that both deflections are small.

APPLICATION TO EXPERIMENTAL RESULTS

In reference 1, Southwell applied his method of analysis to the results of eight column tests made by T. von Kármán and published in 1909. In order to show that the more general equations of the present paper apply equally as well, these same data are reanalyzed in table I and figure 3. The method of least squares was used to establish the best-fitting straight line for each set of data plotted in figure 3. This procedure was used in order that the personal equation would be eliminated in the manner followed by Southwell.

In table II the results of Southwell's analysis and the analysis made in this report are compared. Inspection of the last two columns shows that the analysis made in this report predicts the critical load as closely as the analysis made by Southwell and that both predictions are in close agreement with theory.

The series of equation (1) gives the deflection curve under load P_1. This series also gives the deflection curve when $P_1 = 0$ except that each coefficient a_n has a value different than when $P_1 > 0$. If b_n is substituted for a_n when $P_1 = 0$, the relation between a_n and b_n is given by equation (10) of reference 1, which is, in the notation of this paper,

$$a_n = \frac{b_n}{1 - \dfrac{P_1}{n^2 \, P_{crit}}} \tag{17}$$

From this relation it is concluded that, as P_1 approaches P_{crit}, the first term in the series of equation (1) predominates. Thus a_1 is a close approximation of the deflection y_1 at the middle of the strut.

If the deflections y recorded by von Kármán at the middle of the strut (see table I) represent the true deflections from the condition of zero load, the value of a_1 deduced from the best-fitting line in figure 3 should be in close agreement with the measured value of y_1. Inspection of table II shows that these values of a_1 and y_1 are not always in close agreement. The disagreement is not confined to the short columns but is also present in one of the long columns (strut 2). This fact rules out yielding of the material at high stresses as a possible explanation of the disagreement.

One explanation of the agreement of a_1 and y_1 in table II in some cases and disagreement in other cases is as follows: When a_1 and y_1 agree, the strut was very nearly straight or the initial reading was taken at a very low load. When a_1 and y_1 disagreed, the strut had initial deflection and the initial reading was taken at other than zero load.

This explanation was reached as a result of the following reasoning. There must be some load on the column to hold it in the testing machine. Consequently, if the initial readings are taken at a load greater than zero, the deflections recorded are smaller than they would have been had the initial reading been taken at zero load. Thus, when the value of a_1 deduced from the best-fitting line in figure 3 is in disagreement with the value of y_1 recorded at the middle of the strut, y_1 should always be less than a_1. This conclusion is verified by the values of a_1 and y_1 given in table II except for strut 4b. In this strut a number of the readings at the lower loads are known to be out of line with the rest of the readings. (See fig. 8 of reference 1.)

Another explanation of the disagreement between a_1 and y_1 in table II is as follows: Slight variations in the cross-sectional area and the material properties are possible in any strut. The effect of these variations appears in the values of a_1 and P_{crit} deduced from the best-fitting line in figure 3. On a percentage basis, a_1 is much more sensitive than P_{crit} to variations of the type mentioned.

If b_1 is the value of a_1 when $P_1 = 0$, the values of b_1 that correspond to a_1 are obtained from equation (17) with $n = 1$. These values of b_1 are listed in table II where the values of b_1 deduced by Southwell are also tabulated. Inspection of the values of b_1 (N.A.C.A.) reveals that the largest initial deflections are found in those struts for which a_1 and y_1 disagree. This fact adds weight to the first explanation of the disagreement between a_1 and y_1.

When Southwell estimated the critical load for the struts tested by von Karman, the slopes of the straight lines in figure 8 of reference 1 were determined by experimental points taken near the critical load. According to theory, the deflection approaches infinity when P approaches P_{crit}. Although Southwell intended to plot y/P against y it may be that $(y \pm \Delta)/P$ was plotted against

$y \pm \Delta$, where Δ is a constant error in the measurement of y. When Δ is small and P is near P_{crit}, the estimated value of P_{crit} is very nearly the same in each case. It is therefore to be expected that Southwell's estimate of the critical load should be as good as the estimate made in this report. (See table II.)

CONCLUSIONS

1. For the analysis of experimental observations in problems of elastic stability, it is found that the following equation holds:

$$\frac{y - y_1}{P - P_1} = \frac{y - y_1}{P_{crit} - P_1} + \frac{a_1 \sin \frac{\pi x}{L}}{P_{crit} - P_1}$$

where

P and y are the load and the corresponding deflection, respectively.

P_1 and y_1 are initial values of P and y, respectively.

P_{crit} is the critical value of P.

$a_1 \sin \frac{\pi x}{L}$ is a constant related to y_1.

2. The straight line obtained by plotting $\frac{y - y_1}{P - P_1}$ as ordinate against $y - y_1$ as abscissa cuts the horizontal axis at the distance $\left[a_1 \sin \frac{\pi x}{L} \right]$ from the origin and the inverse slope of the line is $P_{crit} - P_1$.

3. For the experimental data examined, the critical load obtained from the slope of the straight line established by the test data was found to agree well with the theoretical critical load. The values of a_1, however, did not always agree well with the value of y_1 observed

at the middle of the strut, indicating that the initial reading may not have been taken at zero load.

4. It is not always practicable to obtain the initial reading of load and deflection at zero load. For this reason the method described herein for the analysis of experimental observations in problems of elastic stability is more useful than methods previously described in which the initial reading must be at zero load.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., July 20, 1938.

APPENDIX

Derivation of Equation (13) from the Equations Given in Reference 1

If it is assumed that the deflections y_o at zero load are given by the equation

$$y_o = \sum_{n=1}^{n=\infty} b_n \sin \frac{n\pi x}{L} \tag{18}$$

then, according to reference 1, the deflections y at load P are

$$y = \sum_{n=1}^{n=\infty} \frac{b_n}{1 - \dfrac{P}{n^2 P_{crit}}} \sin \frac{n\pi x}{L} \tag{19}$$

The deflections y_1 at load P_1 are therefore

$$y_1 = \sum_{n=1}^{n=\infty} \frac{b_n}{1 - \dfrac{P_1}{n^2 P_{crit}}} \sin \frac{n\pi x}{L} \tag{20}$$

Subtraction of equation (20) from equation (19) gives

$$y - y_1 = \sum_{n=1}^{n=\infty} \left[\frac{1}{1 - \dfrac{P}{n^2 P_{crit}}} - \frac{1}{1 - \dfrac{P_1}{n^2 P_{crit}}} \right] b_n \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{n=\infty} \frac{1}{\dfrac{n^2 P_{crit} - P_1}{P - P_1} - 1} \; \frac{b_n \sin \dfrac{n\pi x}{L}}{1 - \dfrac{P_1}{n^2 P_{crit}}} \tag{21}$$

But

$$a_n = \frac{b_n}{1 - \dfrac{P_1}{n^2 \ P_{crit}}} \tag{17}$$

With this substitution, equation (21) agrees with equation (13)

$$y - y_1 = \sum_{n=1}^{n=\infty} \frac{a_n \sin \dfrac{n\pi x}{L}}{\dfrac{n^2 \ P_{crit} - P_1}{P - P_1} - 1} \tag{13}$$

Deflection at One Load Expressed in Terms

of Deflection at Another Load

If equation (1) gives the deflections at load P_1, equation (13) gives the increase in deflections that result when the load is increased from P_1 to P. Consequently, addition of equations (1) and (13) gives for y, the deflections at load P,

$$y = y_1 + y_2$$

$$= \sum_{n=1}^{n=\infty} \frac{a_n \sin \dfrac{n\pi x}{L}}{1 - \dfrac{P - P_1}{n^2 \ P_{crit} - P_1}}$$

$$= \frac{a_1 \sin \dfrac{\pi x}{L}}{1 - \dfrac{P - P_1}{P_{crit} - P_1}} + \frac{a_2 \sin \dfrac{2\pi x}{L}}{1 - \dfrac{P - P_1}{2^2 \ P_{crit} - P_1}} + \ . \ . \ . \ . \tag{22}$$

As P approaches P_{crit}, the first term in the series of equation (22) predominates. Thus when P approaches

P_{crit}, it is possible to write as an approximation for equation (22)

$$y = \frac{a_1 \sin \frac{\pi x}{L}}{1 - \frac{P - P_1}{P_{crit} - P_1}} \tag{23}$$

If y_1 is accurately given by the first term of the series (1), y_2 and y are accurately given by the first terms of the series (13) and (22), respectively. Consequently, as long as the deflection curve remains a sine curve, the relations given by equations (14), (15), and (23) are exact.

REFERENCES

1. Southwell, R. V.: On the Analysis of Experimental Observations in Problems of Elastic Stability. Proc., Royal Soc., A, vol. 135, 1932, pp. 601-616.

2. Ayrton, W. E., and Perry, John: On Struts. The Engineer, Dec. 10, 1886 pp. 464-465; Dec. 24, 1886, pp. 513-515.

Table I

T. von Kármán's Struts

[Mild Steel: $E = 2,170,000$ kg/cm^2]

Strut	P axial load (kg)	y deflection at middle of strut (mm)	$P - P_1$ (kg)	$y - y_1$ (mm)	$\frac{y - y_1}{P - P_1}$ mm per kg
1	2,260	0.01			
	3,020	.025	760	0.015	0.1974×10^{-4}
	3,170	.04	910	.03	.3297
	3,320	.06	1,060	.05	.4717
	3,470	.09	1,210	.08	.6612
	3,620	.25	1,360	.24	1.765
2	4,520	0.02			
	4,830	.05	310	0.03	0.9677
	5,130	.11	610	.09	1.475
	5,280	.24	760	.22	2.895
	5,430	.86	910	.84	9.231
3a	6,030	0.01			
	7,540	.03	1,510	0.02	0.1325
	8,290	.11	2,260	.10	.4425
	8,520	.52	2,490	.51	2.048
3b	*7,840	0.02			
	8,140	.05			
	8,290	.07	150	0.02	1.333
	8,445	.11	305	.06	1.967
	8,600	.21	460	.16	3.478
4a	*9,050	0.02			
	*9,660	.025			
	10,260	.03			
	10,560	.07	300	0.04	1.333
	10,710	.10	450	.07	1.556
	10,860	.13	600	.10	1.667
	11,010	.25	750	.22	2.933
	11,160	.73	900	.70	7.778
4b	*3,020	0.03			
	*4,530	.05			
	*6,030	.07			
	*7,540	.09			
	*8,300	.12			
	9,050	.15			
	9,805	.23	755	0.08	1.060
	9,960	.26	910	.11	1.209
	10,110	.29	1,060	.14	1.321
	10,260	.33	1,210	.18	1.488
	10,410	.41	1,360	.26	1.912
	10,560	.52	1,510	.37	2.450
	10,710	.71	1,660	.56	3.373
	10,860	1.46	1,810	1.31	7.238
5	*9,050	0.01			
	*10,560	.03			
	10,860	.05			
	11,160	.07	300	0.02	0.6667
	11,470	.10	610	.05	.8197
	11,770	.15	910	.10	1.099
	12,070	.22	1,210	.17	1.405
	12,370	.30	1,510	.25	1.656
	12,520	.45	1,660	.40	2.410
6	*10,560	0.01			
	*12,070	.04			
	12,370	.06			
	12,670	.10	300	0.04	1.333
	12,970	.15	600	.09	1.500
	13,270	.25	900	.19	2.111
	13,430	.34	1,060	.28	2.642
	13,580	.74	1,210	.68	5.620

*The data for these loads were rejected by Southwell on grounds that are stated at the beginning of paragraph 10 in reference 1. Consequently, the lowest load not so rejected is here taken as P_1.

Table II

Summary of Analyses Made of von Kármán's Tests

Strut	y_1 recorded in test (mm)	a_1 deduced from best-fitting line in figure 3 (mm)	b_1 deduced value		P_{crit} (estimated)		P_{crit} given by theoretical formula (kg)	$\frac{P_{crit} \text{ (estimated)}}{P_{crit} \text{ (theoretical)}}$	
			Southwell (mm)	N.A.C.A. (mm)	Southwell (kg)	N.A.C.A. (kg)		Southwell	N.A.C.A.
1	0.01	0.0164	0.005	0.006	3,712	3,711	3,790	0.980	0.980
2	.02	.0602	.005	.011	5,453	5,495	5,475	.995	1.004
3a	.01	.0135	.005	.004	8,590	8,587	8,645	.994	.993
3b	.05	.0679	.005	.005	8,758	8,794	8,610	1.017	1.020
4a	.03	.0821	.003	.007	11,220	11,269	10,980	1.022	1.025
4c	.15	.1217	.030	.022	11,090	11,037	10,920	1.015	1.011
5	.05	.1350	.010	.023	12,815	13,095	12,780	1.003	1.023
6	.06	.1304	.010	.014	13,750	13,833	13,980	.984	.990

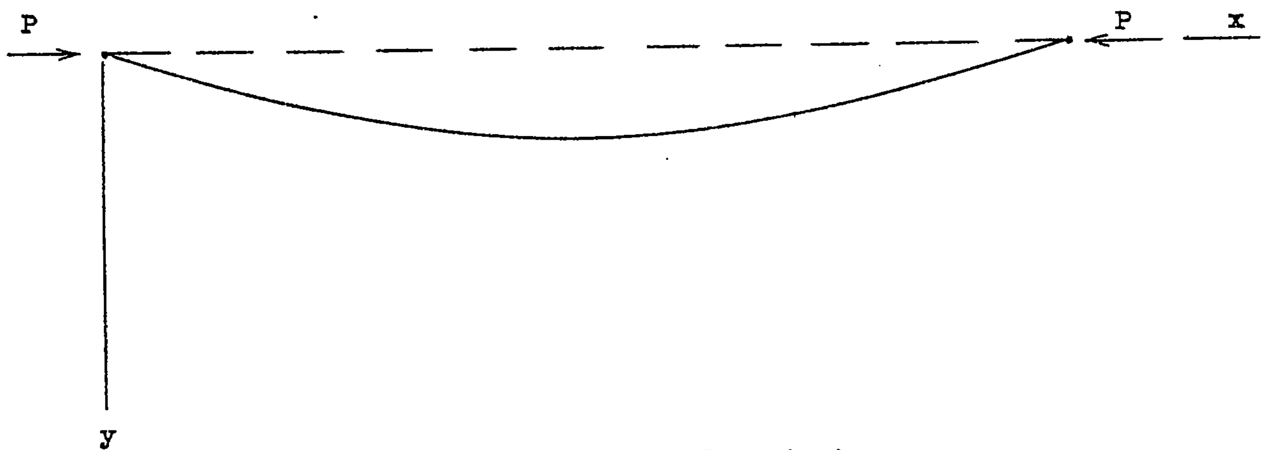

Figure 1.- Euler strut.

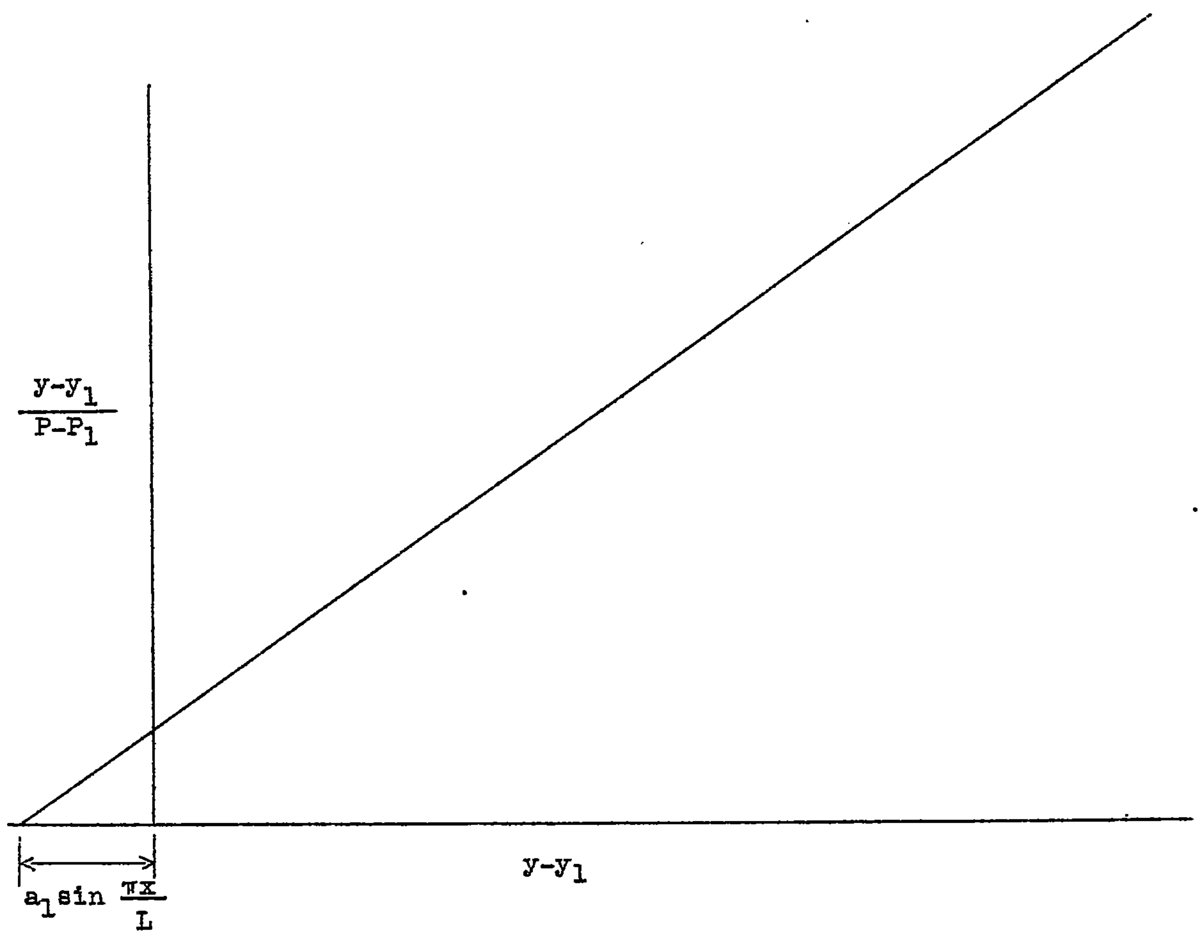

Figure 2.- Graph of equation(15).

Strut	Slenderness ratio	Cross section (mm × mm)	Effective length (mm)
1	175.8	18.2 × 30.1	925.0
2	146.0	18.1 × 30.1	763.0
3a	116.2	18.1 × 30.1	608.0
3b	116.1	18.05 × 30.05	607.5
4a	103.0	18.1 × 30.05	538.0
4b	103.0	18.0 × 30.05	538.0
5	95.3	18.05 × 30.05	497.0
6	91.3	18.1 × 30.05	478.5

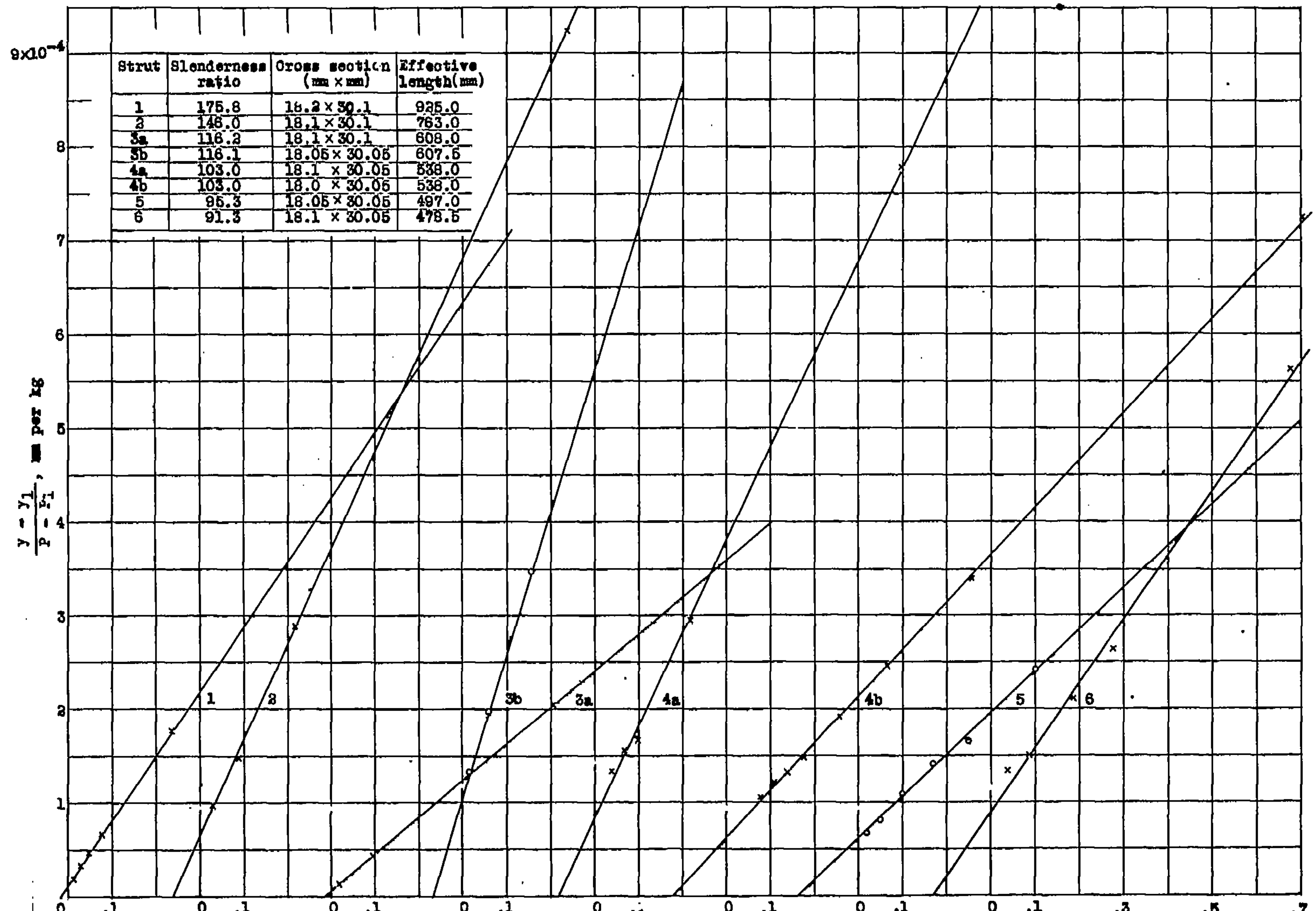

Figure 3.- T. von Kármán's struts (reference 1).